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Robust entangled states

Florian Mintert

Physikalisches Institut, Albert-Ludwigs Universität Freiburg, Hermann-Herder Str. 3, Freiburg, Germany

E-mail: florian.mintert@physik.uni-freiburg.de

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Abstract

We establish a technique to find the states with most robust entanglement in dissipative quantum systems and explicitly construct those states for various environments.

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1. Introduction

The potential to form entangled states is one of the central distinctions between quantum objects and their classical counterparts. Therefore, the stability properties of entanglement teach us a lot about the emergence of classicality in quantum systems of growing size. Besides that—or also, even more importantly—entangled states are the central building block for many promising applications [1] in quantum information theory, so that there is a considerable interest not only in the preparation of highly entangled states, but also in the preservation of entanglement over sufficiently long periods of time that allow us to execute a quantum algorithm or different tasks.

Generally, entanglement decays due to environment coupling [2–4]. However, the decay of entanglement does not necessarily follow that of the density matrices coherences. In particular, since entanglement is *not* a linear function of the underlying quantum state, there can be states whose entanglement is significantly more robust than that of other states.

2. Robust states

Here, we seek those states of a finite dimensional bipartite quantum system whose entanglement is most robust in a given situation of environment coupling. That is, given an entanglement measure E , we look for those states, for which the temporal increment \dot{E} is maximal.

2.1. Target functional

Such tasks require an entanglement measure that can be evaluated or estimated easily. Entanglement measures, however, are rather intricate to evaluate since most of them rely

on a mathematical optimization procedure that can be solved without extensive numerical optimizations only in exceptional cases. What we consider here, is not a measure $E(\rho)$ itself, but rather its time derivative $\dot{E}(\rho, \dot{\rho})$, and the technical difficulties to evaluate the latter are certainly not smaller than for the former.

Therefore, we make use of a purely algebraic estimate of concurrence [5] in terms of a bilinear functional of the density matrix ρ . The concurrence of a *bipartite* pure state $|\Psi\rangle$ can be defined via a spin-flip operation for two-level systems [5], or in terms of a linearized entropy of the reduced density matrix of one of the subsystems [6], or, also, via the expectation value of a suitably chosen operator with respect to the duplicate quantum state $|\Psi\rangle \otimes |\Psi\rangle$ [7] for systems of arbitrary finite dimension. The extension to mixed states can be performed with the help of a convex roof [8] $c(\rho) = \inf \sum_i p_i |\Psi_i\rangle \langle \Psi_i|$, where the infimum is to be found among all pure state decompositions $\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i|$ of the state ρ . Whereas this infimum can be found algebraically only for small systems [9] or some special states [10], it can be bounded from below by

$$c^2(\rho) \geq \text{tr} \rho \otimes \rho V =: \tau(\rho), \quad (1)$$

for general states [11], where $V = P_- \otimes P_- - 1/2(P_- \otimes P_+ + P_+ \otimes P_-)$ is defined in terms of the projectors P_{\mp} on the antisymmetric and symmetric components of the duplicated Hilbert spaces of either subsystem. This is exact for pure states, and provides a very good approximation for weakly mixed states, which makes it an ideal tool for the following investigations where we will consider the decay of initially pure states into mixed ones. With this bound, we can approximate the temporal increment of the tangle as

$$\frac{\partial c^2(\rho)}{\partial t} \simeq \dot{\tau}(\rho, \dot{\rho}) = \text{tr}((\dot{\rho} \otimes \rho + \rho \otimes \dot{\rho})V) = 2 \text{tr}(\dot{\rho} \otimes \rho V), \quad (2)$$

where the last equality is due to the permutation symmetry of V .

In the following, $\dot{\tau}$ in equation (2) will be maximized over pure initial states. Typically, weakly entangled states show more robust entanglement than highly entangled ones, and initially separable states have a vanishing temporal increment under coupling to local environments. In order to ensure that our optimizations will not simply yield separable states as those with most robust entanglement, we will fix the initial tangle $\tau(\Psi) = \tau_0$, and seek those states with given τ_0 that maximize the temporal increment of the tangle

$$\dot{\tau}_{\text{opt}} = \max(\dot{\tau}(\tau, \dot{\rho}_{\Psi}) | \tau(\Psi) = \tau_0). \quad (3)$$

The only precondition for this task is that equations of motion for the system density matrix are given, i.e., system Hamiltonian and environment coupling mechanisms are known.

2.2. Preferred basis

Like any entanglement measure, the tangle is invariant under local unitaries, and given this invariance, any state can be represented in its Schmidt decomposition [12]

$$\sum_i \sqrt{\lambda_i} |\phi_i\rangle \otimes |\bar{\phi}_i\rangle, \quad (4)$$

where all entanglement properties are described in terms of the Schmidt coefficients λ_i , and $\{|\phi_i\rangle\}$ and $\{|\bar{\phi}_i\rangle\}$ are local bases for the individual subsystems, the so-called Schmidt bases. However, the environment coupling typically breaks this invariance for $\dot{\tau}$, i.e. in contrast to τ which is a function of the Schmidt coefficients only, the time derivative $\dot{\tau}$ also depends on the Schmidt bases. Therefore, the present maximization cannot be restricted to a variation of Schmidt coefficients only, but it has to be performed also of all possible choices of Schmidt

bases. Nevertheless, most situations of environment coupling define a special basis, the so-called pointer basis that consists of states that are most robust under environment coupling. Performing numerical optimizations over general states we found that typically states with such pointer bases as Schmidt bases have the most robust entanglement. We will therefore take exactly those states, optimize over their Schmidt coefficients, and convince ourselves at the end, with the comparison to numerical optimizations over general states, that this restriction still allows one to find the states with the most robust entanglement.

Given an initial state in Schmidt decomposition, the most robust entanglement is then determined by the solution of polynomial equations

$$\sum_i \alpha_i \frac{\partial \tau(\Psi)}{\partial \lambda_i} = 0, \quad \text{and} \quad \tau(\Psi) = \tau_0, \quad (5)$$

where the prefactors α_i need to satisfy $\sum_i \alpha_i \frac{\partial \tau(\Psi)}{\partial \lambda_i} = 0$ to ensure that the variation over the λ_i is performed over states with a constant tangle. Such sets of polynomial equations can very reliably and efficiently be solved with the help of Groebner bases [13]. The risk of finding local instead of global minima that always affects numerical optimization is basically dispelled.

2.3. Environment models

To show our presently derived techniques to identify robust states at work we will focus in the following on some exemplary situations of environment coupling. Specifically, we will consider a description in terms of a Master equation [14–16] $\dot{\rho} = \sum_i \mathcal{L}_i \rho$, with a Lindbladian \mathcal{L}_Σ that acts like

$$\mathcal{L}_i \rho = \gamma (2 \Sigma_i \rho \Sigma_i^\dagger - \Sigma_i^\dagger \Sigma_i \rho - \rho \Sigma_i^\dagger \Sigma_i). \quad (6)$$

For coupling operators Σ we consider the case of *local* environments, where any such quantity acts nontrivially only on a single subsystem, since this corresponds to the typical situation in which the subsystems are macroscopically separated. In this case the two-particle operator Σ_1 is of the form

$$\Sigma_1 = \sigma \otimes \mathbb{1}, \quad (7)$$

with a coupling operator σ that describes the effective influence of the environment coupling on the first subsystem, and analogously $\Sigma_2 = \mathbb{1} \otimes \sigma$ for the second subsystem.

The two iconic models for evolutions of quantum states in quantum information processing are the ‘amplitude damping’ channel that corresponds to $\sigma = \sigma_- = (\sigma_x - i\sigma_y)/2$, and the ‘phase damping’ channel that corresponds to $\sigma = \sigma_z$ [17] in terms of the Pauli spin operators. Here, we consider generalizations of these situations to quantum systems with more than just two levels. Typical generalizations to higher dimensional systems are often derived for the harmonic oscillator, where the generalization of σ_- is the annihilation operator a , and the generalization of σ_z is the number operator $n = a^\dagger a$. This, however, is a rather specific situation that cannot describe general experimental situations sufficiently well. For example, there could be several excited states that have comparable life time, or rather different ones that are incompatible with those resulting from the harmonic oscillator operators. We therefore take the generalization

$$\sigma_{\text{dc}}(q) = \sum_i (i+1)^q |i\rangle\langle i|, \quad (8)$$

of the harmonic oscillator case to describe decoherence. Here, there is the variable parameter ‘ q ’ that allows us to change the properties of the dephasing mechanism. The harmonic oscillator case is recovered for $q = 1$, but changing the value of q allows us to adjust the decay

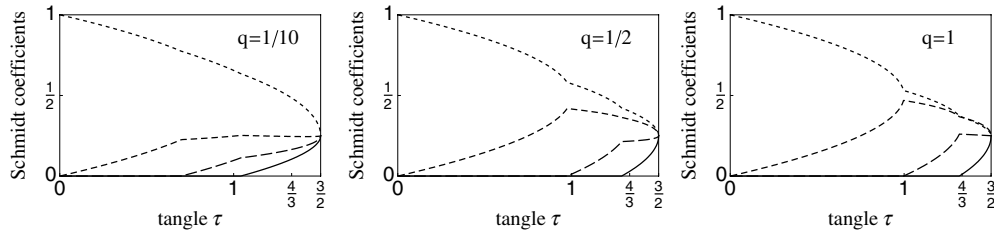


Figure 1. Schmidt coefficients of the states with the most robust entanglement against spontaneous decay as a function of tangle τ . The different insets correspond to different values of the parameter q that characterizes the decay mechanism in equation (9). The Schmidt coefficients are distinguished by their dashing with increasing length of dashing from λ_0 (shortest dashing) to λ_3 (solid line).

rates for the coherences between different levels. As a generalization of spontaneous decay, we take

$$\sigma_{\text{ad}}(q) = \sum_i (i + 1)^q |0\rangle\langle i + 1|, \tag{9}$$

that is the case where all excited levels decay to the ground state with adjustable decay rates. This is different from harmonic oscillator case, where excited states decay to the ground state via all lower lying states, but that it rather resembles the situation of an atom or ion with several excited states that decay directly to the ground state.

Obviously, also these models cannot provide an exhaustive description of very general dissipative dynamics. However, these two situations will provide a good insight in the interplay of stability of entanglement and environment coupling, and situations with specific differences as compared to equations (8) and (9) can easily be investigated along the lines that we present in the following.

Figure 1 shows the Schmidt coefficients of the states with the most robust tangle against environment coupling as given in equation (9) together with equation (6) as a function of tangle τ for different values of the parameter q . For tangles larger than 1, necessarily three Schmidt coefficients need to be positive, and values larger than $4/3$ require at least four positive Schmidt coefficients. The right-most inset of figure 1 shows a situation with significantly different decay times for the individual Schmidt-basis states. Third and fourth basis states are occupied only for a value of τ above the threshold values $\tau = 1$ and $\tau = 4/3$. This is different in the case of comparable life times of the individual basis states as depicted in the left-most inset. Here, the third and fourth levels are occupied already for significantly smaller values of τ . That is, even if a given value of the tangle can be realized also with only two occupied levels, it is actually favorable to occupy more levels, and thereby enhance the stability of entanglement. This enhancement can be attributed to a larger occupation of the stable ground state, as it can be seen in figure 1: whenever an additional level gets occupied, there is a kink in the largest Schmidt coefficient that shows how the decrease of the population of the stable levels is slowed down. Indeed, if enough excited levels are available, it is possible to realize a given amount of entanglement with an occupation of the stable ground state that is arbitrarily close to unity; and if the decay times of the excited states are comparable, then this is exactly what yields the most robust states. If, however, the decay times of the excited states are more broadly distributed, then it is actually beneficial to avoid excitation of fragile levels, which results in a state with few finite Schmidt coefficients.

The situation is quite different in the case of decoherence that is displayed in figure 2. For values of q that are smaller than 1, it is indeed favorable to occupy the levels $|2\rangle$ and $|3\rangle$, since

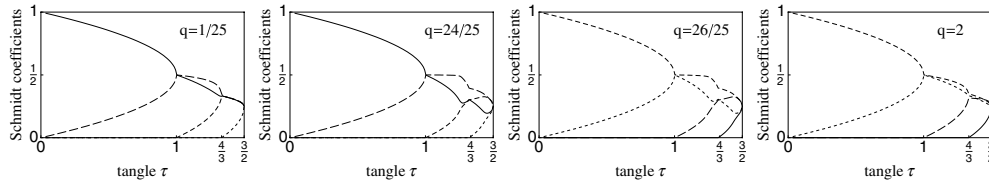


Figure 2. Schmidt coefficients of the states with most robust entanglement against decoherence analogous to figure 1. The different insets correspond to different values of q defined in equation (8).

the decoherence times for superpositions of the states $|2\rangle \otimes |2\rangle$ and $|3\rangle \otimes |3\rangle$ are longest. If q is greater than 1, the situation changes and superpositions of the states $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$ are most stable so that there is a qualitative change between the two insets that correspond to the values $q = 24/25$ and $q = 26/25$. The third and fourth levels are occupied only above the threshold values $\tau = 1$ and $\tau = 4/3$ for all values of q . However, the distribution of Schmidt coefficients at a given tangle $\tau > 1$, depends on q rather strongly: whereas the large Schmidt coefficients are of comparable size while there is a single small Schmidt coefficient for $q \ll 1$ and for $q \gg 1$, it is more favorable to have more broadly distributed Schmidt coefficients for $q \sim 1$. Although the interpretation of Schmidt coefficients is qualitatively different from those in figure 1, basic reasoning in the interpretation is rather similar to the one above: for values of q that differ strongly for 1 (i.e. $q \ll 1$ and $q \gg 1$) there are two states whose coherent superposition dephases significantly slower than the coherent superposition of any other pair of levels. Therefore, it is favorable to have potentially large amplitudes for these states, and additional levels are excited only to the extent that is necessary to realize a given amount of entanglement. Since the impact of dephasing depends only on the magnitude of the density matrices' coherence, but not on the occupations, it is indeed favorable to have nearly equally large occupation of the levels whose coherent superposition is rather stable, since this allows us to reach rather high entanglement with comparatively small excitation of fragile levels. For $q \simeq 1$ the situation is rather different. Now, there is not such a preferred pair (or group) of levels whose coherent superposition dephases most slowly. However, there is one state (the ground state for $q < 1$ and the fourth state for $q > 1$) whose coherent superpositions decay slightly slower (at given difference in the excitation number) than those of other states. Therefore, in this regime states that have a rather large amplitude for these states have favorable dynamical properties, and the potentially large occupation of the least fragile levels gives rise to the rather broad distribution of Schmidt coefficients.

2.4. Numerical confirmation

So far, we have been assuming that the optimal states have their Schmidt bases given in terms of the environment-induced pointer bases. Figure 3 shows a comparison of the algebraic solutions obtained under this assumption with numerical solutions, where the optimization has been performed over general initial states. The left inset displays the logarithm $\log(-\dot{\tau})$ of the temporal increment $\dot{\tau}$ of the tangle for different values of q . Apparently, all numerically obtained data points lie above the algebraic solutions that are shown as lines. This implies that no improvement in the optimization can be obtained dropping the assumption on the Schmidt bases. Besides that, there are also several data points that lie significantly above the algebraic solutions. These points correspond to local maxima that pose a serious challenge to numerical optimizations, that, however, are hardly a problem for our algebraic solutions. The right

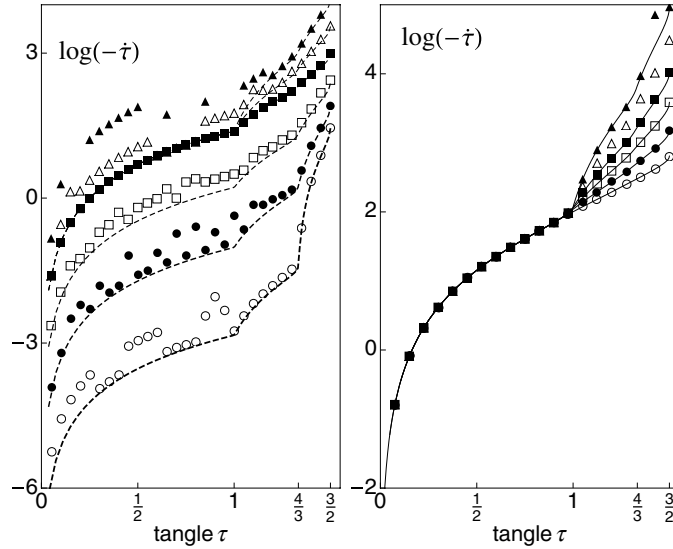


Figure 3. Comparison of numerically optimized temporal increment of the tangle and optimized $\dot{\tau}$ with pointer bases as Schmidt bases: the left inset corresponds to the situation of dephasing and the right inset to spontaneous decay. The circles, squares and triangles display the numerical solutions for $q = 1/4, 1/2, 3/4, 1, 5/4$ and $3/2$ (from bottom to top), and the lines show the algebraically obtained solutions as a function of the parameter q defined in equations (9) and (8).

inset of figure 3 shows the analogous situation for decay. Here, the numerical optimization is significantly more reliable than in the case of decoherence, and there is only one data point that corresponds to a local maximum. Thus, the conjecture that Schmidt and pointer bases coincide is justified by the data shown.

2.5. Long-time behavior

Finally, let us compare the optimal solutions that have been obtained with the approximation equation (2) with the time dependence of the state that is not subject to this approximation. Here, we use the bounds [18, 19] that are known to provide a very accurate estimate of concurrence, in particular for weakly mixed states, which is exactly the situation that we are dealing with here. The black line in the left plot of figure 4 shows the time evolution of τ for the state with initial tangle $\tau_0 = 7/3$ for decoherence with $q = 1$. The gray lines show the tangle for 1000 randomly chosen states with the same initial tangle. The tangle of the optimized case clearly decays significantly slower than that of the other states, and the optimized states carry sizable entanglement at times where most other states have become separable. The left plot of figure 4 shows the analogous situation for decay with $q = 1/10$ and initial states with initial tangle $\tau_0 = 8/3$. Here, we have been choosing a very small value of q , since in this case the different excited states have comparable decay times so that the formation of a pointer basis is not very pronounced. Therefore, this example is a rather challenging test for our approach that assumes the existence of such a basis. In contrast to the left plot, there is no gap between the time evolutions of the random states and the optimized one, and the inset that shows a zoom for the short time behavior indicates that there are a few states that have a slightly more robust entanglement than the state that was found ideal. However, the differences between the tangle of the optimized state and the tangle of the most

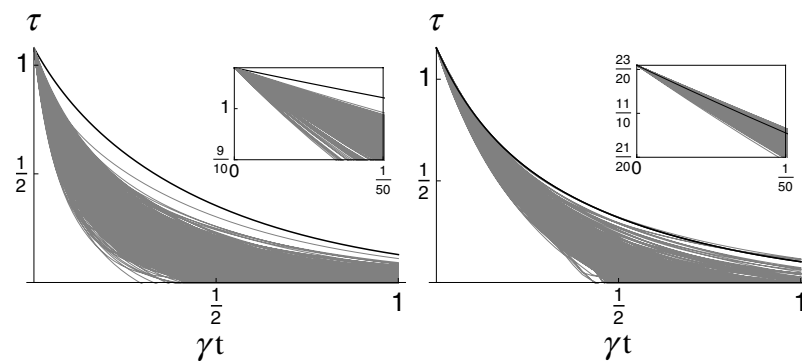


Figure 4. Time dependence of tangle for the obtained most robust states (black line) and 1000 randomly chosen initial states with a given initial tangle τ_0 . The left plot corresponds to decoherence (equation (8)) with $\tau_0 = 7/3$, the right plot to decay (equation (9)) with $\tau_0 = 8/3$.

robust random states is within the error margins of the different estimates of a mixed state tangle.

Another feature that strikes the eye here is the fact that in both insets of figure 4 the tangle of the initially optimized states turns out to be most robust over all times, although the optimization has only been performed for the initial time step. Whereas this is a mere observation here, a more rigorous footing for the generality of such behavior—that would exceed the scope of the present letter by far—will facilitate the search for robust mixed states tremendously. In particular, prior observations that the dynamics of entanglement [20] is often characterized very well by the first infinitesimal time step strongly substantiate our conjecture that optimal mixed states are actually given by the decay products of initially pure states.

3. Outlook

Here, we have been focusing on bipartite system. The same ideas of maximizing the temporal increment of tangle can also be applied to multipartite systems for which equation (1) can be generalized [21]. The investigation for bipartite systems was facilitated with the Schmidt decomposition. For general multipartite systems, however, there is not a simple generalization of this tool, and there is an abundance of different classes of multipartite entangled states [22, 23] that draws a very intransparent picture of entangled states. Similar to the bipartite case, where the Schmidt decomposition could be reproduced via the most robust states under very general situations of environment coupling, we expect that similar investigations on multipartite systems will allow us to identify robust classes of entangled states and provide a significantly more physical characterization of multi-partite states than a mere distinction of mathematical classes can provide.

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